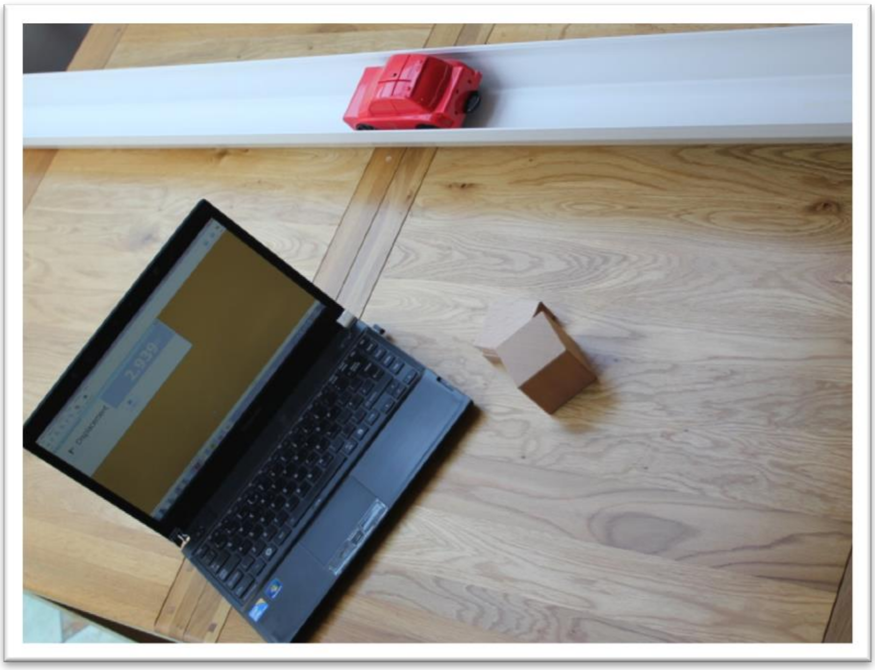
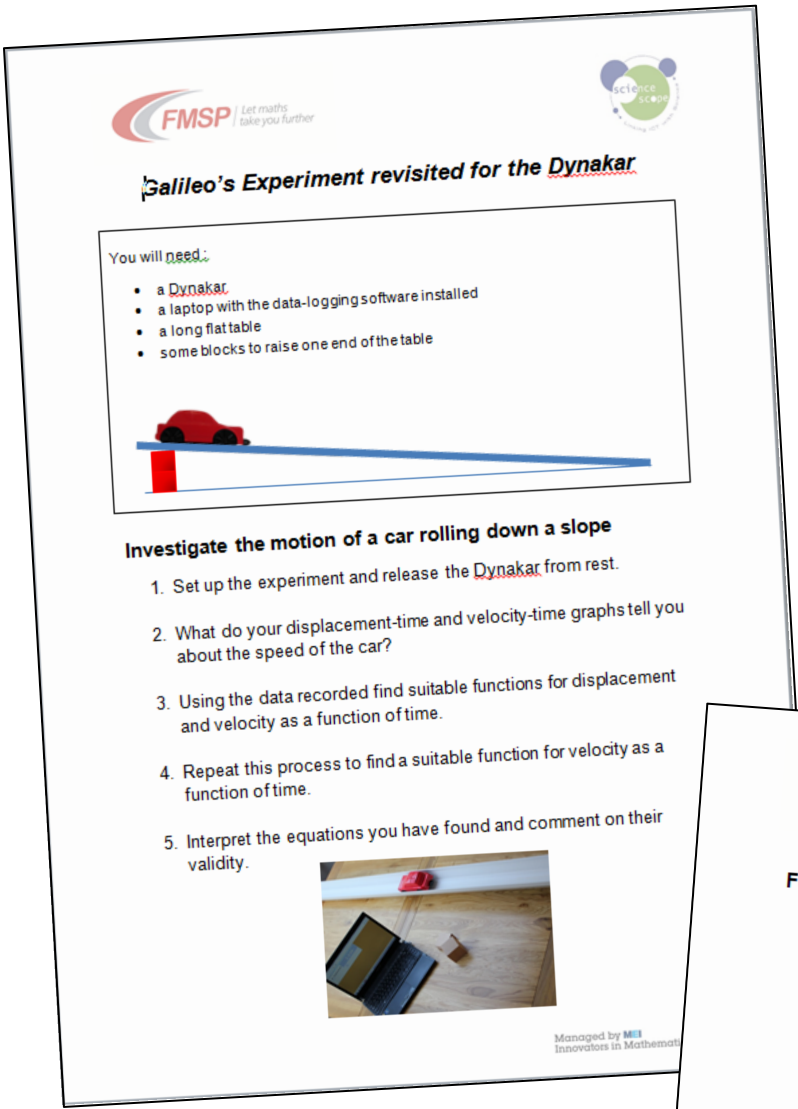
Galileo's Investigation



Teachers' Guide to using the Dynakar and modelling motion

The activity worksheets



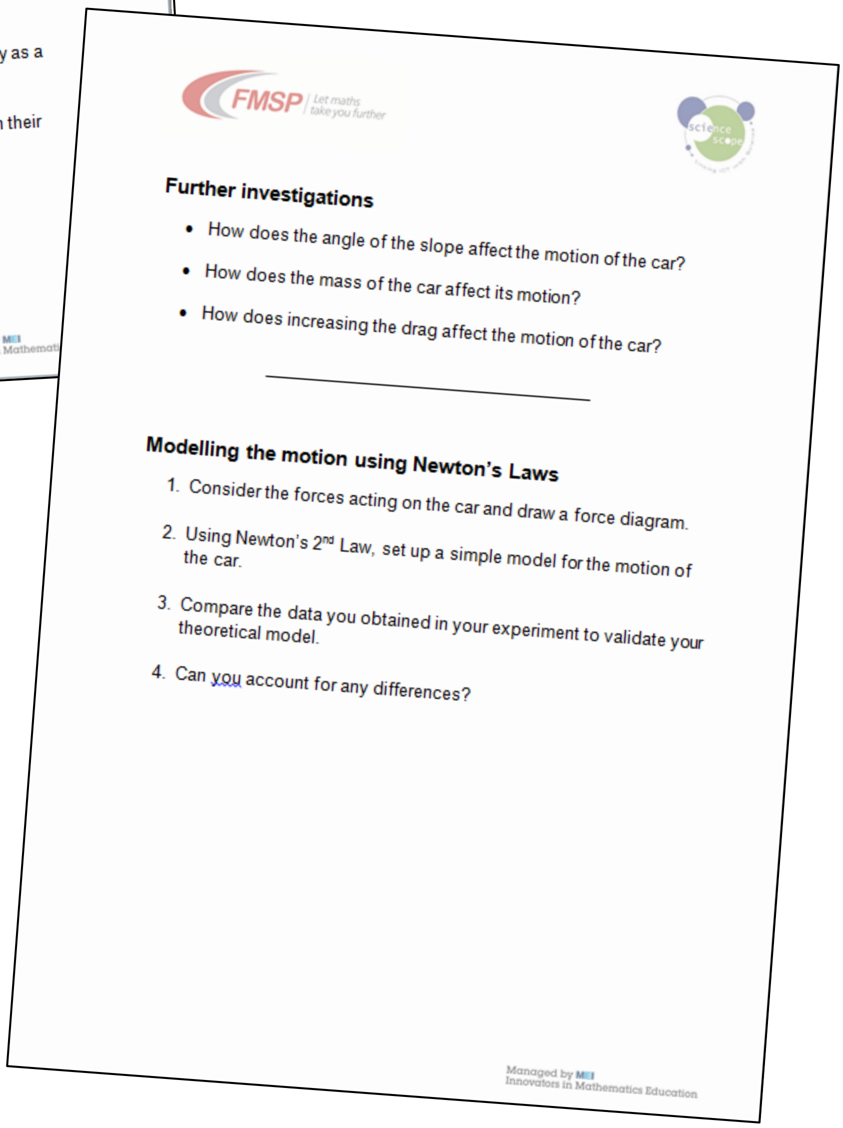
Galileo's Experiment revisited for the Dynakar

You will need :

- a Dynakar
- a laptop with the data-logging software installed
- a long flat table
- some blocks to raise one end of the table

Investigate the motion of a car rolling down a slope

1. Set up the experiment and release the Dynakar from rest.
2. What do your displacement-time and velocity-time graphs tell you about the speed of the car?
3. Using the data recorded find suitable functions for displacement and velocity as a function of time.
4. Repeat this process to find a suitable function for velocity as a function of time.
5. Interpret the equations you have found and comment on their validity.



Further investigations

- How does the angle of the slope affect the motion of the car?
- How does the mass of the car affect its motion?
- How does increasing the drag affect the motion of the car?

Modelling the motion using Newton's Laws

1. Consider the forces acting on the car and draw a force diagram.
2. Using Newton's 2nd Law, set up a simple model for the motion of the car.
3. Compare the data you obtained in your experiment to validate your theoretical model.
4. Can you account for any differences?

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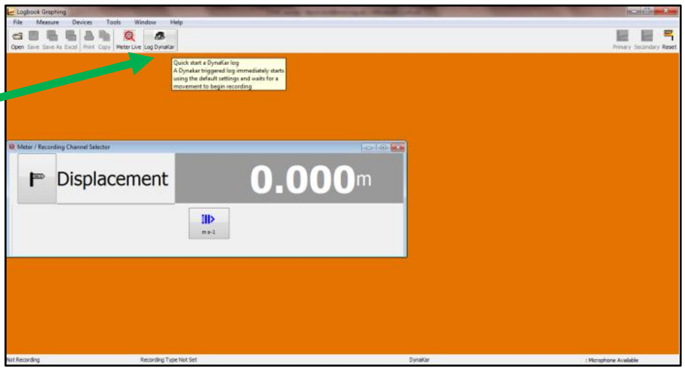
Teachers' Guide to Galileo's Experiment

Setting up the Dynakar and data-logging software

After setting up the Dynakar experiment, open the ***Logbook Graphing*** software.

Check the car's bluetooth dongle is inserted in the PC/laptop and the car is switched on.

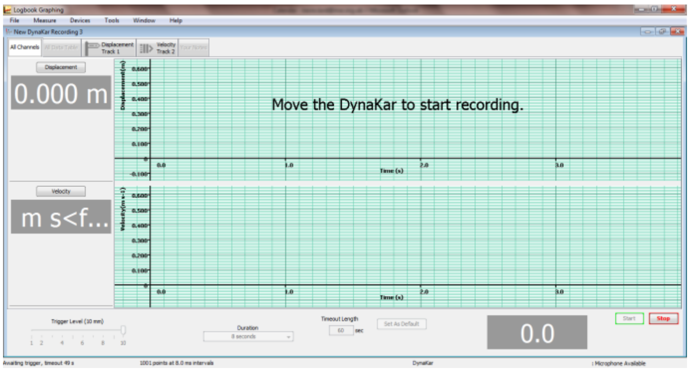
Select Log Dynakar.



You are given 2 options:

- **Quick Start** (data collection begins automatically when car starts moving and for a default period of 8s)
- **Regular Start** (data collection started manually and duration & trigger can be set)

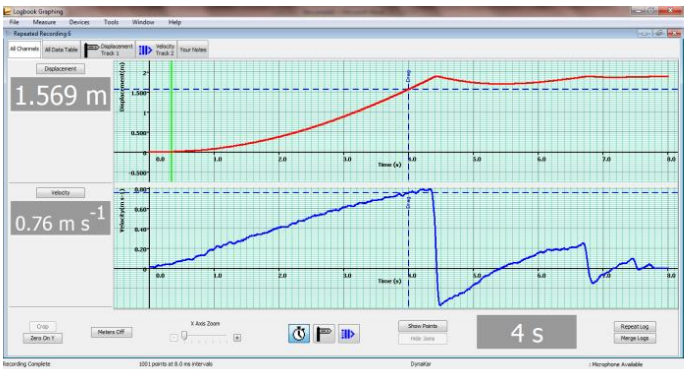
Select Quick Start and the logbook graphing page will open. Check that the displacement is set to 0.000 m.



Release the Dynakar.

The data is captured automatically. After 8s elapses the data collection stops and the graph for velocity-time is smoothed.

From this screen you can observe the displacement-time graph and below it the velocity-time graph. It is possible to identify where and when the Dynakar reached the end of the track and rebounded etc.

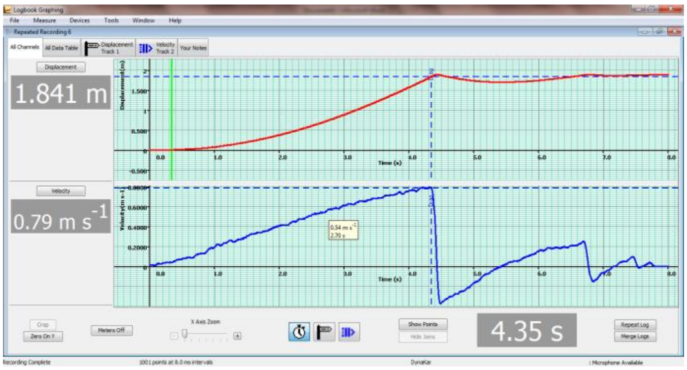


By clicking on any point on the curves, the dashed lines will move and you can read off the time, displacement and velocity at that point.

By hovering on a point you will also see the approximate velocity or acceleration at that particular time.

Click on a point at the end of the motion required.

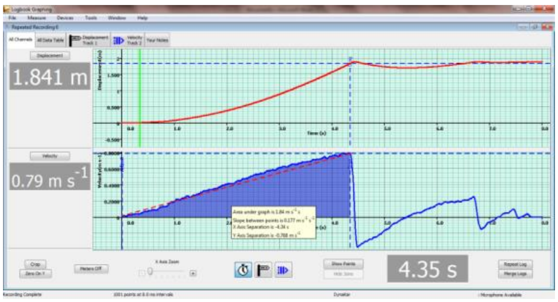
(In this case when the velocity stopped increasing, after 4.35s.)

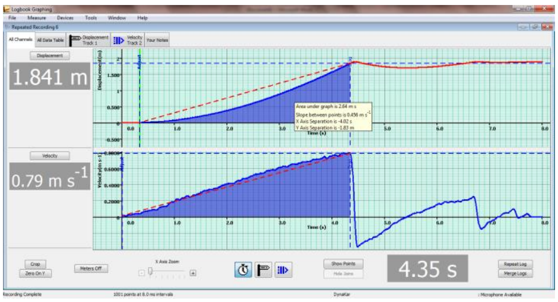


Click and drag the vertical dashed line across to the origin. You will see the area under the curve between these two vertical lines is now shaded.

Hover the cursor over this area and you will be able to read off values for the selected area.

- Average acceleration (gradient of line between the end points)
- Displacement (area under the curve)





Finally with the relevant area highlighted click on Crop. This removes data collected after the car reached the end of the track.

The graphs of displacement-time and velocity-time can be analysed using these tools in the Logbook Graphing software and students should be encourage to discuss and interpret the shape of the curves and any interesting features.

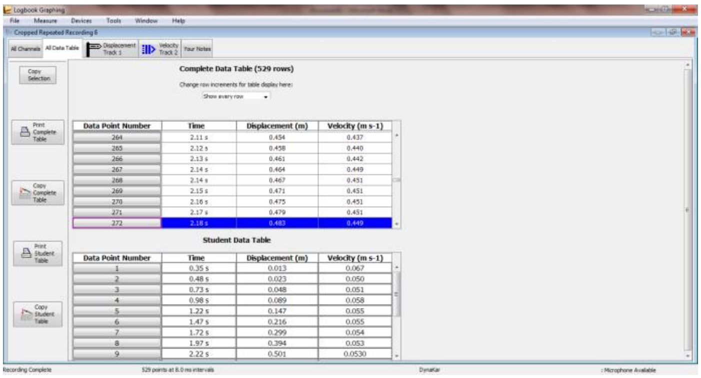
However to model these curves with functions you will need to export the data to a spreadsheet and/or graphing application.

Modelling the motion of the Dynakar

Once the graphs have been cropped so that only the relevant section is displayed, **select "All Data Table"** from the menu buttons at the top of the screen.

This displays two tables of data. (Ignore the lower Student Data Table).

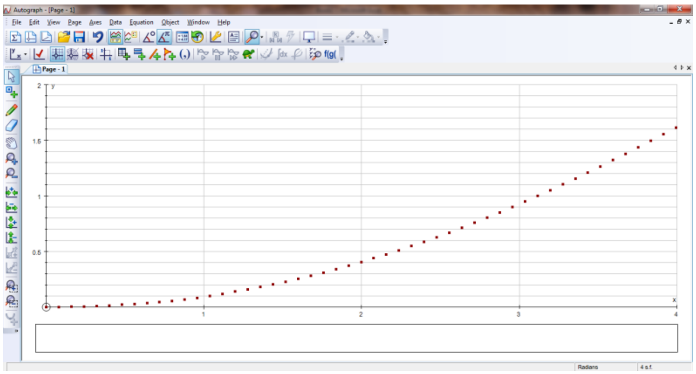
The first table shows all the data points for motion recorded at intervals of approximately 0.01s and consequently there are hundreds of data points.

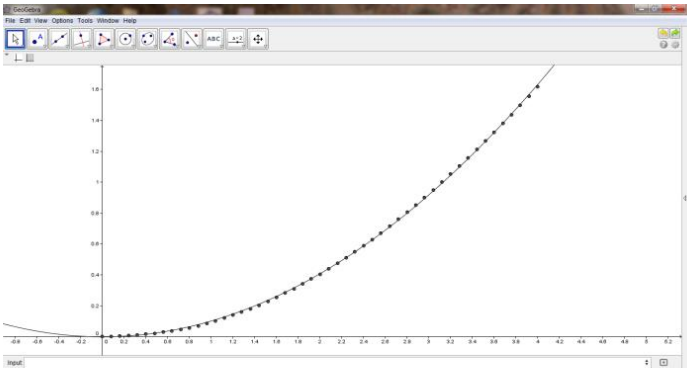


From the drop down menu above the table **select Show every 10th row or Show every 20th row** to reduce the number of values.

Select Copy Complete Table button in the left-hand margin. You can now paste the table into a spreadsheet, which allows you to select the data you wish to graph.

To plot the displacement-time graph, select the time and displacement data and paste into graphing software. For example, in **Autograph**, select ***Data*** and then ***Enter XY Data Set…*** to input the time and displacement data as X and Y coordinates. Plotting the data shows the points with automatically chosen axes.





Alternatively the data set can be entered as a list in Geogebra and analysed.

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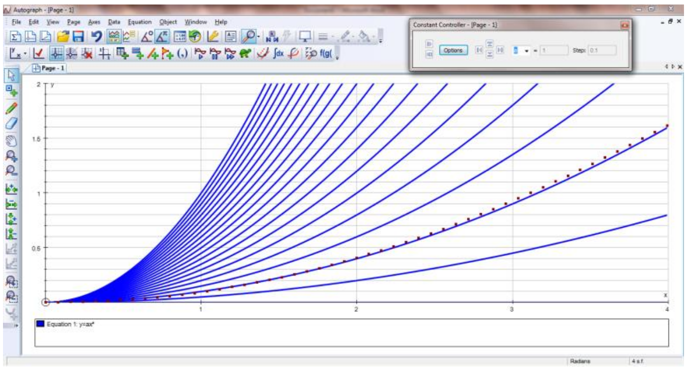
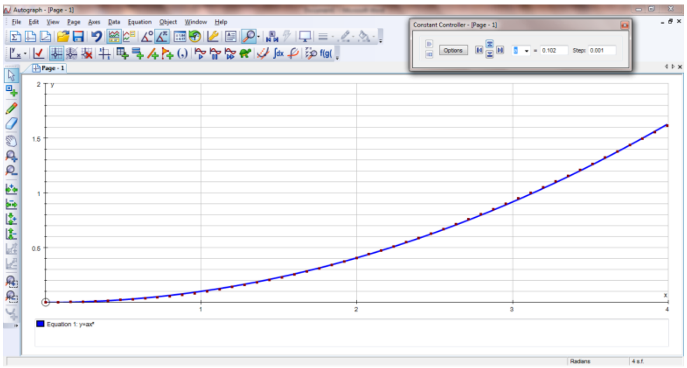
Fitting a function to the data can be done by trial and improvement or by using the animated features of the graphing software. In Autograph this would be the Constant Controller and in Geogebra by using a slider.

Students should consider suitable functions that might fit the data and the interpretation of these functions in the context of the observed motion. For example, a linear function used to model the data for displacement-time may fit the points reasonably well but would imply that the displacement was changing at a constant rate, i.e. the car was moving with a constant velocity. However it should be observed that the car started from rest and then later was moving, indicating its velocity had changed!

Using a quadratic function passing through the origin, e.g. $\mathbf{y = ax^2}$, where **a** is constant to be determined should give the correct general shape.



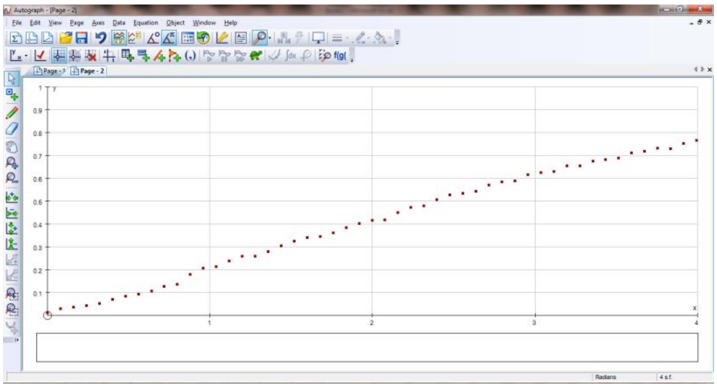
The value of **a** can be found using the constant controller in Autograph either by displaying the family of curves for various values of **a**, or by changing manually the value of **a** until the curve is a good fit for the points.



The function fitting the data shown in this example is $\mathbf{y = 0.102x^2}$

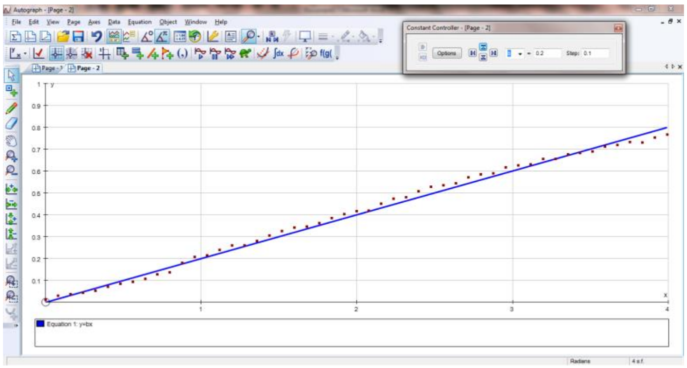
(or $s= 0.102\ t^2$).

The velocity-time graph can be similarly plotted by selecting the data from the spreadsheet. (Note: the velocity data is derived from the recorded displacement data and as such there may be slight errors in the values.)



The function fitting the data shown in this example is **y = 0.2x**

(or v = 2t).



Thus we have models for the motion of the Dynakar rolling down a slight incline,

Displacement	**$s = 0.102\ t^2$**
Velocity	**$v = 2t$.**

Questions for discussion

Students should interpret these formulae and consider their validity.

- If the car was to roll further would the graphs be similar shapes. Are the formulae valid for times greater than 4s?
- What do the formulae predict the velocity or displacement is at a particular time?
- What is the acceleration of the car? Is it constant?

For A level Mechanics students

- How do these experimental values compare with the theoretical values predicted using Newton’s Laws of Motion?
- What other forces may be acting? Can the magnitude of these forces by estimated?

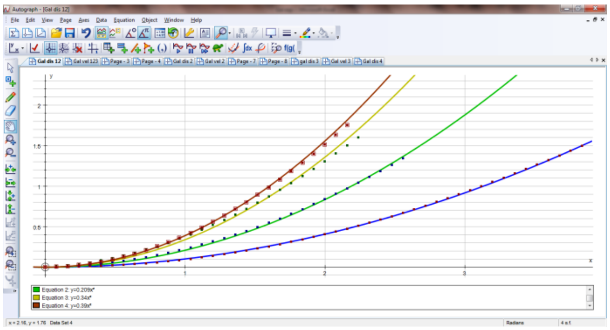
1. Investigating the height of the incline

The experiment can repeated after increasing the height of the incline.

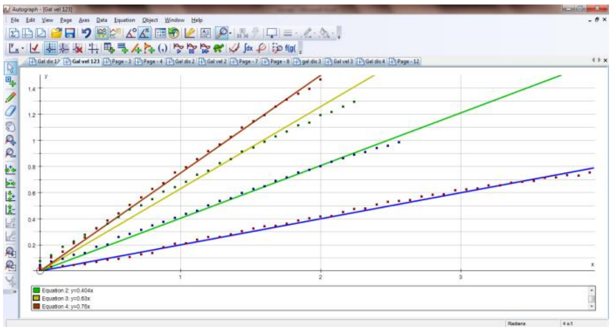
The data and graphs captured by the Logbook Graphing software are sufficient for students to observe that:

- The displacement-time and velocity-time graphs are similar in shape to the previous run, so the motion is similar.
- The car takes less time to cover the distance – hence it is travelling faster.
- The gradient of the velocity-time graph is greater – hence the acceleration is greater.
- The acceleration can be read off the screen by highlighting a section of the velocity-time graph. (The average acceleration should be approximately equal to the value found by modelling the data using a graphing package)

The following screenshots show the displacement-time and velocity-time graphs for four different inclines.



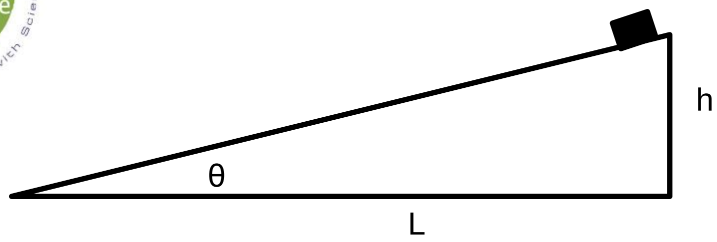
displacement-time graph



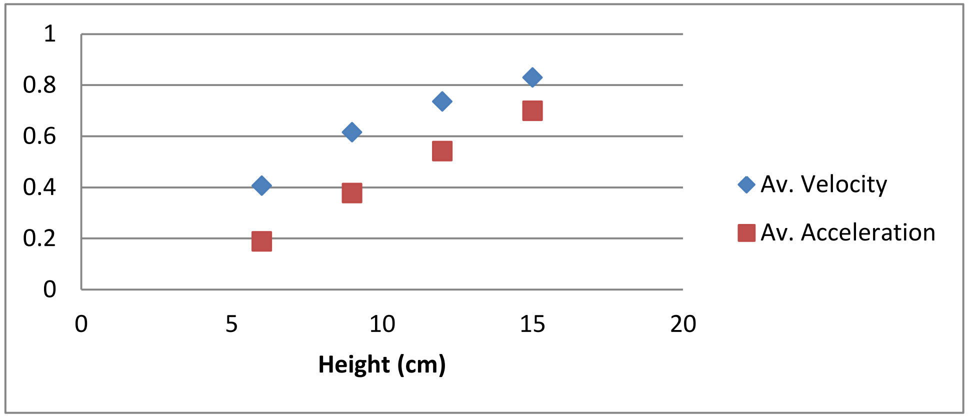
velocity-time graph

The data collected by the Dynakar is very consistent and facilitates detailed modelling in this way. It is clear that the acceleration increases as the angle of the incline increases.

Sample results



			Data from Dynakar				Experimental Model from Autograph	Theoretical Model using Newton's Laws
Height h (cm)	Length L (cm)	Angle θ (rad)	Displacement (m)	Time Taken (s)	Average Velocity (ms^{-1})	Average Acceleration (ms^{-2})	**Acceleration (ms^{-2})**	**Acceleration (ms^{-2})**
6	155	0.039	1.62	4.00	0.405	0.187	**0.20**	**0.38**
9	155	0.058	1.68	2.73	0.614	0.376	**0.39**	**0.57**
12	155	0.077	1.69	2.30	0.734	0.540	**0.63**	**0.76**
15	155	0.096	1.71	2.11	0.828	0.698	**0.75**	**0.94**

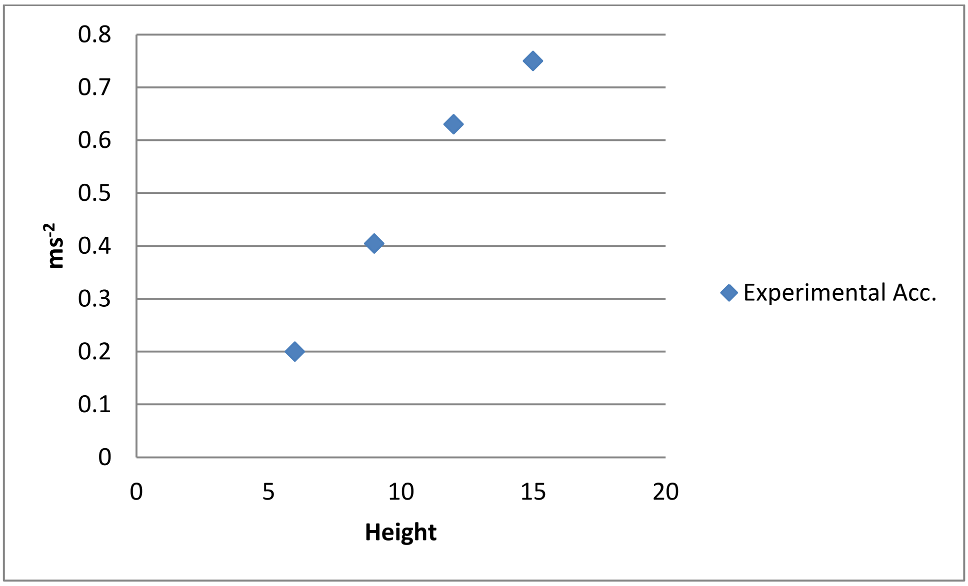


The data that can be obtained easily from the Logbook Graphing software for the Dynakar shows that as the angle increases the average velocity and acceleration increase.

By graphing these results it suggests that there might be a linear relationship between average acceleration and height, where **av acc = 0.057h – 0.14**. Thus when 0.057h = 0.14, h = 2.5cm, there would be no acceleration and the car would remain at rest.

Modelling the experimental data using Autograph

The values for acceleration which can be found by fitting curves to the displacement-time and velocity-time graphs are consistent with the value for the average acceleration obtained from the Logbook Graphing software.



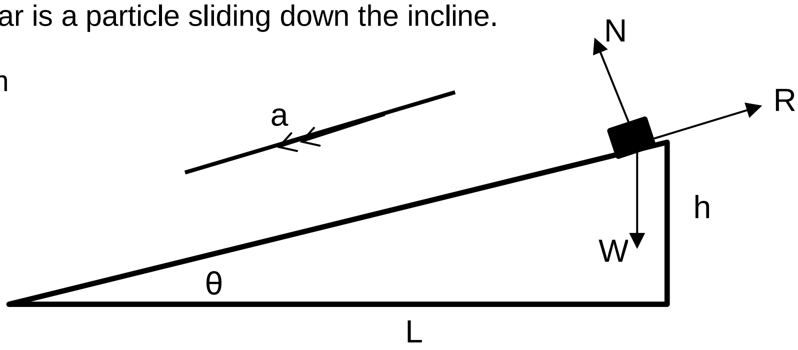
The relationship appears linear, **exp. acc. = 0.063h – 0.16**

Again the experimental acceleration is zero when 0.063h = 0.16, i.e. when h = 2.5 cm.

Modelling the motion using Newton's Laws of Motion

Assume the car is a particle sliding down the incline.

Force diagram



(W = mg is the weight of the Dynakar, R is a constant resistance force, and N is the normal contact force.)

Using Newton's 2nd Law parallel to the incline,

$$W\sin\theta - R = ma$$

$$mg\sin\theta - R = ma$$

If R = 0, then $$g\sin\theta = a$$

How does the theoretical model compare with the experimental model?

The theoretical acceleration (assuming R = 0) is a = $g\sin\theta$.

Angle of incline θ (rad)	**Experimental Acceleration (ms^{-2})**
0.039	**0.20**
0.058	**0.39**
0.077	**0.63**
0.096	**0.75**

Using these values we can estimate values for **g** = a/ sinθ. The values are not constant and are lower than 9.8 ms^{-2} which we would expect.

Taking **g** as 9.8 ms^{-2} in the formula for the acceleration a = $g\sin\theta$ we find that the theoretical acceleration is higher than that found experimentally by approximately 0.18ms^{-2}.

The conclusion is that the resistance force R is not equal to zero.

Therefore a refined model for the acceleration is $a = g\sin\theta - R/m$

We can estimate that R/m is between 0.15 and 0.18 as this is the difference between the observed acceleration and $g\sin\theta$.

If the mass of the car = 0.225g, then the resistance force is $0.034 < R < 0.041$ newtons.

Comparing the theoretical model with the experimental data

Experimental model (from graphing) $a_E = 0.06h - 0.16$

Theoretical model (from Newton's Laws) $a_T = g\sin\theta \; - \; R/m$

Substituting $R/m = 0.16$, $g = 9.8$ and $\sin\theta = \frac{h}{\sqrt{h^2+L^2}} = \frac{h}{\sqrt{h^2+155^2}} \approx \frac{h}{155}$

gives $a_T = 9.8\,\frac{h}{155} - 0.16$

Therefore $a_T = 0.063h - 0.16$

These models are remarkable similar.

2. Investigating the mass of the Dynakar

The incline height was kept constant at 9cm (or an angle of 0.058 rad) and the experiment was repeated increasing the mass of the car by 40g each time.

The acceleration of the car was taken from the Logbook Graphing software. For consistency the measurements were taken 3 times for each different mass.

Sample results for Dynakar

Additional Mass (kg)	Acceleration (ms^{-2}) Run 1	Acceleration (ms^{-2}) Run 2	Acceleration (ms^{-2}) Run 3	Median Acceleration (ms^{-2})
0	0.38	0.37	0.37	**0.37**
0.04	0.37	0.35	0.36	**0.36**
0.08	0.37	0.37	0.38	**0.37**
0.12	0.36	0.38	0.38	**0.38**
0.16	0.38	0.38	0.37	**0.38**

The results indicate that increasing the mass of the Dynakar does not change its acceleration.

Using Newton's 2nd Law you can show that the acceleration **$a = g\sin\theta$**

Conclusion

The acceleration is independent of the mass of the car.

Galileo showed that the acceleration of a falling object is independent of its mass in the 17th century.

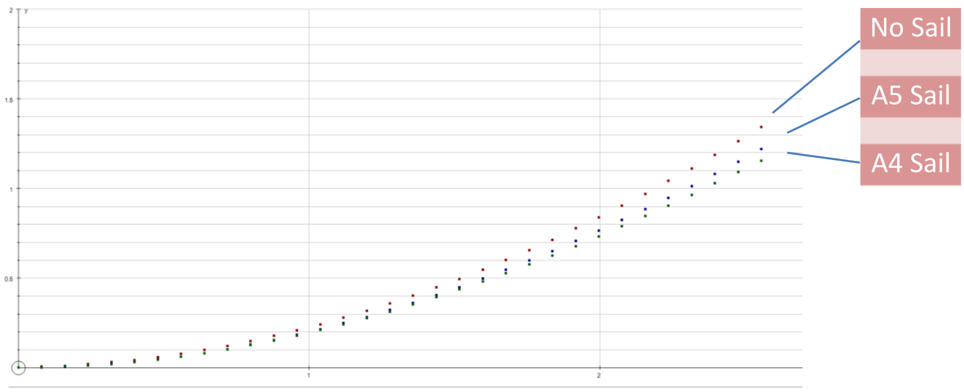
3. Investigating the drag of the Dynakar

The incline height and mass of the car are kept constant at 9cm (or an angle of 0.058 rad) and 225g respectively and the experiment was repeated increasing the drag of the car by using cardboard sails.

Sample results for Dynakar

Area of Sail (cm^2)	Acceleration (ms^{-2}) Run 1	Acceleration (ms^{-2}) Run 2	Acceleration (ms^{-2}) Run 3	Median Acceleration (ms^{-2})
0	0.38	0.37	0.37	**0.37**
315 (15x21)	0.38	0.33	0.31	**0.33**
630 (30x21)	0.28	0.28	0.27	**0.28**
945 (45x21)	0.26	0.25	0.27	**0.26**

The data for displacement time from 3 runs with different sails is below.



The data in the table and the graphs show that increasing the drag has an effect on the acceleration of the Dynakar. The higher the drag, the lower the acceleration.

If the drag is sufficient it may be possible to see that the car is travelling at a constant velocity for final seconds of the motion. This would indicate that the car had reached its terminal velocity. On the displacement-time graphs this is shown by the almost linear sections after 2s.

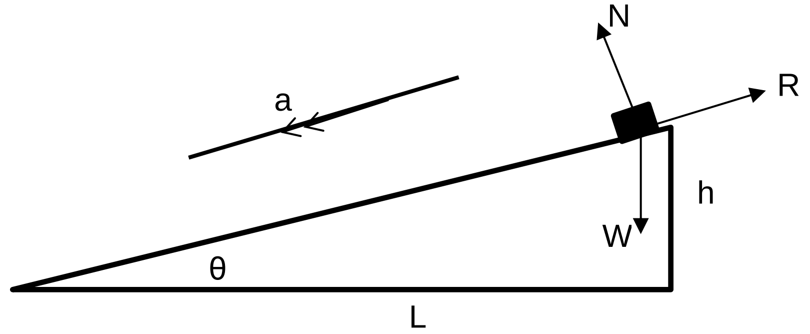
The displacement-time data can be modelled again using quadratic functions, but a refined model would incorporate a drag component.

Modelling the drag force

Two models are commonly used for resistance force, drag. These are:

$$\boldsymbol{R = Kv} \text{ and } \boldsymbol{R = kv^2}$$

Force diagram



where W = mg is the weight of the Dynakar, R is the drag force, and N is the normal contact force.

Using Newton's 2nd Law parallel to the incline,

$$W sin\theta \;-\; R \;=\; ma$$

$$mg sin\theta \;-\; R \;=\; ma$$

Using $\boldsymbol{R = Kv}$ model for the drag force

Newton's 2nd Law gives $\quad ma = mg sin\theta - Kv$

When the car reaches terminal velocity the acceleration is zero. Therefore if *V* is the terminal velocity of the car we have that

$$KV = mg sin\theta$$

To find the velocity and displacement of the car as functions of time requires the integration of the equation for a.

The equation for acceleration can be written as $\quad \frac{dv}{dt} = g sin\theta - \frac{K}{m} v$

Substituting for K : $\quad \frac{dv}{dt} = g sin\theta - \frac{g sin\theta}{V} v = g sin\theta(1 - \frac{v}{V})$

Separating variables: $\quad \int \frac{dv}{1-\frac{v}{V}} = \int g sin\theta\, dt$

$$-V\, ln\left|1 - \frac{v}{V}\right| = g sin\theta t + constant$$

Initial conditions are $t = 0,\ v = 0,$ therefore the $constant = 0$

$$-V\,ln\left|1-\frac{v}{V}\right| = gsin\theta t$$

Rearranging for v:

$$ln\left|1-\frac{v}{V}\right| = -\frac{g}{V}sin\theta\text{t}$$

$$1-\frac{v}{V} = e^{-\frac{g}{V}sin\theta t}$$

The velocity of the car is $\boxed{v = V(1-e^{-\frac{g}{V}sin\theta t})}$

Integrating the expression for v $\quad x = \int V(1-e^{-\frac{g}{V}sin\theta t})dt$

$$x = V(t + \frac{V}{gsin\theta}e^{-\frac{g}{V}sin\theta t}) + constant$$

Initial conditions are $t = 0,\ x = 0.$

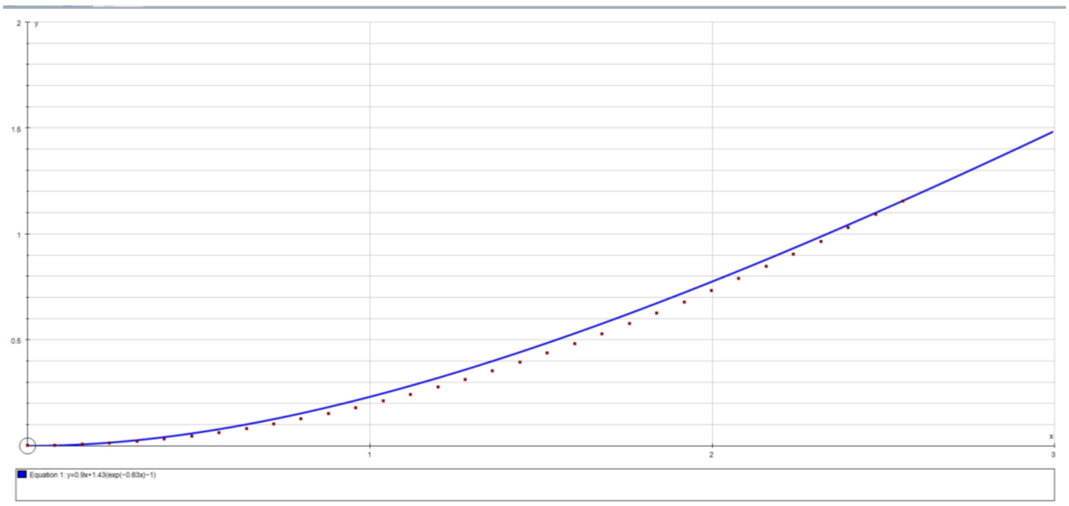
Therefore $\quad 0 = 0 + \frac{V^2}{gsin\theta} + constant$

Hence $\quad \boxed{x = Vt + \frac{V^2}{gsin\theta}(e^{-\frac{g}{V}sin\theta t} - 1)}$

How does the theoretical model compare with the experimental results?

Using an estimate of 0.9ms^{-1} for terminal velocity for sail of size A4 and substituting for g and sinθ into the formula for x gives

$$x = 0.9t + 1.43(e^{-0.63t} - 1)$$



This function is quite a good fit to the data collected.

Alternatively, using $R = Kv^2$ model for the drag force

Newton's 2nd Law gives $ma = mgsin\theta - Kv^2$

When the car reaches terminal velocity the acceleration is zero. Therefore if V is the terminal velocity of the car we have that

$$KV^2 = mgsin\theta$$

To find the velocity and displacement of the car as functions of time requires the integration of the equation for a.

The equation for acceleration can be written as $\frac{dv}{dt} = gsin\theta - \frac{K}{m}v^2$

Substituting for K : $\frac{dv}{dt} = gsin\theta - \frac{gsin\theta}{V^2}v^2 = gsin\theta(1 - \frac{v^2}{V^2})$

Separating variables: $\int \frac{dv}{1-\frac{v^2}{V^2}} = \int gsin\theta\, dt$

Simplifying the denominator on RHS: $\int \frac{V^2}{V^2-v^2} dv = \int \frac{V^2}{(V-v)(V+v)} dv = \int gsin\theta\, dt$

Using partial fractions: $\int \frac{V^2}{(V-v)(V+v)} dv = \int \frac{V}{2(V-v)} + \frac{V}{2(V+v)}\, dv = \int gsin\theta\, dt$

$$\frac{V}{2}\, ln\left|\frac{V+v}{V-v}\right| = gsin\theta t + constant$$

Initial conditions are $t = 0,\ v = 0.$

Therefore $constant = 0$

$$\frac{V}{2}\, ln\left|\frac{V+v}{V-v}\right| = gsin\theta t$$

Rearranging for v: $ln\left|\frac{V+v}{V-v}\right| = \frac{2g}{V} sin\theta t$

$\frac{V+v}{V-v} = e^{\frac{2g}{V}sin\theta t}$ therefore $\frac{V-v}{V+v} = e^{\frac{-2g}{V}sin\theta t}$

The velocity of the car is $\boxed{v = V\frac{(1-e^{\frac{-2g}{V}sin\theta t})}{(1+e^{\frac{-2g}{V}sin\theta t})}}$

For displacement, integrate the expression for v

$$\frac{dx}{dt} = V\frac{(1-e^{\frac{-2g}{V}sin\theta t})}{(1+e^{\frac{-2g}{V}sin\theta t})} = V\frac{(e^{\frac{2g}{V}sin\theta t}-1)}{(e^{\frac{2g}{V}sin\theta t}+1)} = V(\frac{2e^{\frac{2g}{V}sin\theta t}}{(e^{\frac{2g}{V}sin\theta t}+1)} - 1)$$

$$x = V(\frac{2}{\frac{2g}{V}sin\theta}ln\left|e^{\frac{2g}{V}sin\theta t} + 1\right| - t) + constant$$

Initial conditions are $t = 0,\ x = 0$.

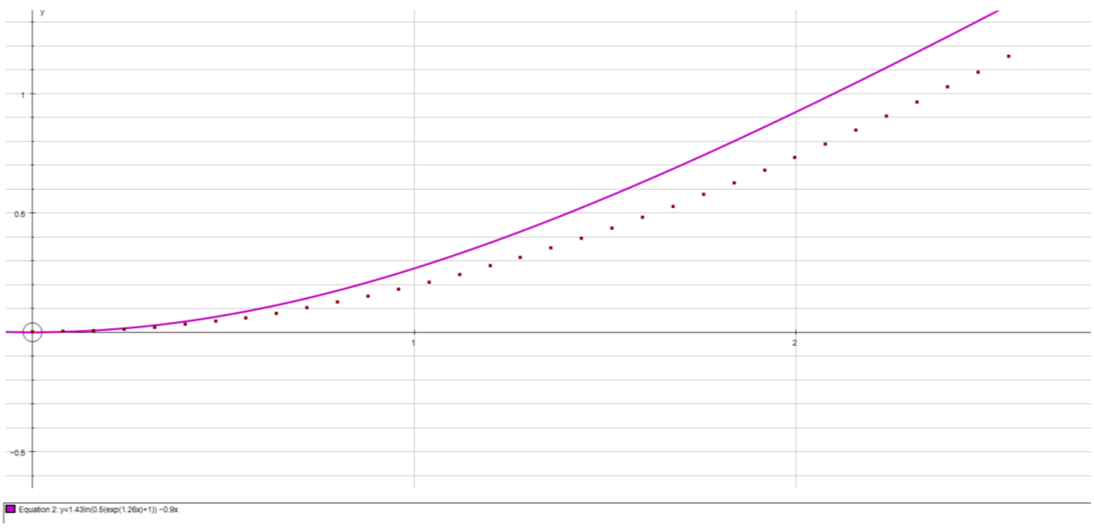
Therefore $$0 = \frac{V^2}{gsin\theta}ln2 - 0 + constant$$

Hence $$\boxed{x = \frac{V^2}{gsin\theta}\ ln\frac{\left|e^{\frac{2g}{V}sin\theta t}+1\right|}{2} - Vt}$$

How does the theoretical model compare with the experimental results?

Using an estimate of 0.9ms^{-1} for terminal velocity for sail of size A4 and substituting for g and sinθ into the formula for x gives

$$x = 1.43\ln\frac{e^{1.26t}+1}{2} - 0.9t$$



This function is not quite such a good fit to the data collected.